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Stability of linear shear flows in shallow water

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The stability or instability of various linear shear flows in shallow water is considered. The linearized equations for waves on the surface of each flow are solved exactly in terms of known special functions. For unbounded shear flows, the exact reflection and transmission coefficients R and T for waves incident on the flow, are found. They are shown to satisfy the relation $|R|^2 = 1 + |T|^2$, which proves that over-reflection occurs at all wavenumbers. For flow bounded by a rigid wall, R is found. The poles of R yield the eigenvalue equation from which the unstable modes can be found. For flow in a channel, with two rigid walls, the eigenvalue equation for the modes is obtained. The results are compared with previous numerical results.

1. Introduction

A shear flow in the x-direction with any velocity profile $U(y)$ is a possible steady motion in shallow water of constant depth. Its stability can be determined from the dispersion equation for small-amplitude surface waves superposed upon it, or equivalently from the coefficients of reflection and transmission of such waves by the flow. These waves also represent the stable and unstable modes of oscillation of the flow. In the special case of linear shear flow of bounded or unbounded extent, the instability was investigated by Satomura (1981*a,b*) and by Takehiro & Hayashi (1992). The latter authors also considered reflection by the flow and showed that over-reflection occurred. Both of these investigations were based upon numerical solution of the governing ordinary differential equation.

We shall solve the problems of stability and reflection for linear shear flows analytically by solving the governing ordinary differential equation exactly in terms of known special functions. From the solution we shall obtain the dispersion equation and the reflection and transmission coefficients. The virtue of the solution is that it yields exact results for all parameter values. Therefore it can be used as a check on numerical and asymptotic methods, and it extends them beyond the parameter ranges which they cover.

One could also solve these problems by asymptotic methods for waves short compared to the width of the shear flow. The value of the asymptotic method is that it can be applied to any current profile $U(y)$, and it is valid for small wavelengths, which cannot be treated numerically. We have used this method previously (Knessl & Keller 1992) to treat the instability of a shear flow on a rotating sphere in the equatorial β-plane approximation. It verified and extended the previous numerical results of Griffiths, Killworth & Stern (1982) and Hayashi & Young (1987). We had intended to use the asymptotic method on linear shear flows before we found the exact solution.

2. Formulation

Let a small-amplitude surface wave of height $\mathrm{Re}[h(y)\mathrm{e}^{-\mathrm{i}k(x-ct)}]$, with $k > 0$, be superposed on a shear flow with velocity $U(y)$ in the x-direction in shallow water of constant depth. Here y is a transverse horizontal coordinate, and not the vertical coordinate, as in some early studies of flows with vertical shear. Then $h(y)$ satisfies the following dimensionless ordinary differential equation, in which F is the Froude number based upon the undisturbed depth (Satomura 1981*a*):

$$h'' - 2U'(U-c)^{-1}h' + k^2[F^2(U-c)^2 - 1]h = 0. \tag{2.1}$$

For the linear shear flow $U(y) = y$, (2.1) is

$$h'' - 2(y-c)^{-1}h' + k^2[F^2(y-c)^2 - 1]h = 0. \tag{2.2}$$

To simplify (2.2) we introduce the new variables ξ and $H(\xi)$ defined by

$$\xi = kF(y-c)^2, \qquad h(y) = H(\xi). \tag{2.3}$$

Then (2.2) becomes

$$H_{\xi\xi} - \frac{1}{2\xi}H_{\xi} + \left(\frac{1}{4} - \frac{k}{4F\xi}\right)H = 0. \tag{2.4}$$

We can transform this equation into a standard form by writing

$$\xi = \mathrm{i}\eta, \quad H(\xi) = \xi^{1/4}G(\mathrm{i}\eta). \tag{2.5}$$

Then from (2.4) and (2.5) we get

$$G_{\eta\eta} + \left(-\frac{1}{4} - \frac{\mathrm{i}k}{4F\eta} - \frac{5}{16\eta^2}\right)G = 0. \tag{2.6}$$

Equation (2.6) is Whittaker's equation (Abramowitz & Stegun 1970, p. 505, eq. 13.1.31) with $\kappa = -\mathrm{i}k/4F$ and $\mu = \pm 3/4$.

The general solution of (2.6) can be written in terms of Kummer's function $M(\frac{1}{2} + \mu - \kappa, 1 + 2\mu, \eta)$ (Abramowitz & Stegun, eqs. 13.1.2, 13.1.32) with arbitrary constants $\tilde{A}$ and $\tilde{B}$:

$$G(\eta) = \tilde{A}\eta^{5/4}\mathrm{e}^{-\eta/2}M\left(\frac{5}{4} + \frac{\mathrm{i}k}{4F}, \frac{5}{2}, \eta\right) + \tilde{B}\eta^{-1/4}\mathrm{e}^{-\eta/2}M\left(-\frac{1}{4} + \frac{\mathrm{i}k}{4F}, -\frac{1}{2}, \eta\right). \tag{2.7}$$

Upon using (2.7) in (2.5) we obtain $H(\xi)$, which by (2.3) is equal to $h(y)$. Thus we have, with new constants A and B,

$$h(y) = H(\xi) = A\xi^{3/2}\mathrm{e}^{\mathrm{i}\xi/2}M\left(\frac{5}{4} + \frac{\mathrm{i}k}{4F}, \frac{5}{2}, -\mathrm{i}\xi\right) + B\mathrm{e}^{\mathrm{i}\xi/2}M\left(-\frac{1}{4} + \frac{\mathrm{i}k}{4F}, -\frac{1}{2}, -\mathrm{i}\xi\right). \tag{2.8}$$

This is the exact general solution of (2.2).

3. Reflection and transmission coefficients for an unbounded flow

We shall now use (2.8) to determine the reflection and transmission coefficients R and T for a wave incident from $y = -\infty$, when the shear flow extends from $y = -\infty$ to $y = +\infty$. To do so we must consider the asymptotic forms of $h(y)$ as y tends to $\pm\infty$. From (2.3) we see that as $y \to +\infty$ it follows that $\xi \to +\infty$, since k and F are positive. Therefore $\arg(-\mathrm{i}\xi) = -\pi/2$. Similarly, as $y \to -\infty$, we see that $\xi \to (+\infty)\mathrm{e}^{2\pi \mathrm{i}}$ and therefore $\arg(-\mathrm{i}\xi) = 3\pi/2$. For these two values of $\arg(-\mathrm{i}\xi)$ the asymptotic forms of M are given by eq. 13.5.1 on p. 508 of Abramowitz & Stegun (1970). We use the leading term for M to obtain

$$\begin{aligned} h(y) &= H(\xi) \\ &\sim \pm A\Gamma\left(\tfrac{5}{2}\right)\left[\frac{\mathrm{e}^{\mathrm{i}\xi/2}\mathrm{e}^{-\mathrm{i}\pi/2(5/4+\mathrm{i}k/4F)}\xi^{1/4-\mathrm{i}k/4F}}{\Gamma(\frac{5}{4}-\mathrm{i}k/4F)} + \frac{\mathrm{e}^{-\mathrm{i}\xi/2}\mathrm{e}^{\mathrm{i}\pi/2(5/4-\mathrm{i}k/4F)}\xi^{1/4+\mathrm{i}k/4F}}{\Gamma(\frac{5}{4}+\mathrm{i}k/4F)}\right] \\ &\quad + B\Gamma\left(-\tfrac{1}{2}\right)\left[\frac{\mathrm{e}^{\mathrm{i}\xi/2}\mathrm{e}^{-\mathrm{i}\pi/2(-1/4+\mathrm{i}k/4F)}\xi^{1/4-\mathrm{i}k/4F}}{\Gamma(-\frac{1}{4}-\mathrm{i}k/4F)} + \frac{\mathrm{e}^{-\mathrm{i}\xi/2}\mathrm{e}^{\mathrm{i}\pi/2(-1/4-\mathrm{i}k/4F)}\xi^{1/4+\mathrm{i}k/4F}}{\Gamma(-\frac{1}{4}+\mathrm{i}k/4F)}\right], \\ &\qquad y \to \pm\infty. \end{aligned} \tag{3.1}$$

These two asymptotic forms of $h(y)$ differ only in the sign in front of A.

We note that $h(y)$ is the sum of two waves with phases $\pm\xi/2 = \pm kF(y-c)^2/2$. We have written the solution with the time factor $\mathrm{e}^{+\mathrm{i}kct}$. Therefore the outgoing waves are the ones for which the phase decreases as $|y-c|$ increases, i.e. the ones with phase $-\xi/2$ at both $\pm\infty$. The waves with phase $+\xi/2$ are incoming. When a wave is incident from $y = -\infty$ and no wave is incident from $y = +\infty$, the coefficient of the incoming wave $\mathrm{e}^{\mathrm{i}\xi/2}$ must vanish as $y \to +\infty$. By setting this coefficient equal to zero in (3.1) with the upper sign, which holds as $y \to +\infty$, we obtain

$$\frac{A\Gamma(\frac{5}{2})\mathrm{e}^{-5\pi\mathrm{i}/8}}{\Gamma(\frac{5}{4}-\mathrm{i}k/4F)} + \frac{B\Gamma(-\frac{1}{2})\mathrm{e}^{\mathrm{i}\pi/8}}{\Gamma(-\frac{1}{4}-\mathrm{i}k/4F)} = 0. \tag{3.2}$$

We can use (3.2) to write A in terms of B.

The reflection coefficient R is defined to be the ratio of the amplitude of the outgoing or reflected wave $\mathrm{e}^{-\mathrm{i}\xi/2}$ at $y = -\infty$ to the amplitude of the incoming or incident wave $\mathrm{e}^{\mathrm{i}\xi/2}$ at $y = -\infty$. These two coefficients are contained in (3.1) at $y = -\infty$, and they give

$$R = \frac{\dfrac{-A\Gamma(\frac{5}{2})\mathrm{e}^{-5\pi\mathrm{i}/8}}{\Gamma(\frac{5}{4}-\mathrm{i}k/4F)} + \dfrac{B\Gamma(-\frac{1}{2})\mathrm{e}^{\mathrm{i}\pi/8}}{\Gamma(-\frac{1}{4}-\mathrm{i}k/4F)}}{\dfrac{-A\Gamma(\frac{5}{2})\mathrm{e}^{5\pi\mathrm{i}/8}}{\Gamma(\frac{5}{4}+\mathrm{i}k/4F)} + \dfrac{B\Gamma(-\frac{1}{2})\mathrm{e}^{-\mathrm{i}\pi/8}}{\Gamma(-\frac{1}{4}+\mathrm{i}k/4F)}} = \frac{2\mathrm{e}^{\mathrm{i}\pi/4}}{-\mathrm{i}\dfrac{\Gamma(\frac{5}{4}-\mathrm{i}k/4F)}{\Gamma(\frac{5}{4}+\mathrm{i}k/4F)} + \dfrac{\Gamma(-\frac{1}{4}-\mathrm{i}k/4F)}{\Gamma(-\frac{1}{4}+\mathrm{i}k/4F)}}. \tag{3.3}$$

The last expression in (3.3) is obtained by using (3.2) to eliminate A/B. Next we simplify the last denominator in (3.3) by using the relation $[\Gamma(z)\Gamma(1-z)]^{-1} = \pi^{-1}\sin\pi z$, and we obtain

$$R = \frac{-2\pi\mathrm{i}\mathrm{e}^{-\pi k/4F}}{\Gamma(\frac{5}{4}-\mathrm{i}k/4F)\Gamma(-\frac{1}{4}-\mathrm{i}k/4F)}. \tag{3.4}$$

To rewrite R in a more useful form we first set $K = k/4F$. Then we can write the denominator in (3.4) as follows, using eq. 6.1.32 on p. 256 of Abramowitz &

Stegun (1970):

$$\Gamma\left(\tfrac{5}{4}-\mathrm{i}K\right)\Gamma\left(-\tfrac{1}{4}-\mathrm{i}K\right)=\left(\tfrac{1}{4}-\mathrm{i}K\right)\Gamma\left(\tfrac{1}{4}-\mathrm{i}K\right)\frac{\Gamma\left(\tfrac{3}{4}-\mathrm{i}K\right)\Gamma\left(\tfrac{1}{4}+\mathrm{i}K\right)}{\left(-\tfrac{1}{4}-\mathrm{i}K\right)\Gamma\left(\tfrac{1}{4}+\mathrm{i}K\right)}$$

$$=\frac{\left(\tfrac{1}{4}-\mathrm{i}K\right)\Gamma\left(\tfrac{1}{4}-\mathrm{i}K\right)}{\left(-\tfrac{1}{4}-\mathrm{i}K\right)\Gamma\left(\tfrac{1}{4}+\mathrm{i}K\right)}\frac{\pi\sqrt{2}}{\cosh\pi K+i\sinh\pi K}. \tag{3.5}$$

By using (3.5) in (3.4) and simplifying, we get

$$R=\left(1-\mathrm{i}\mathrm{e}^{-\pi k/2F}\right)\mathrm{e}^{-\mathrm{i}\pi/4}\left(\frac{-1+\mathrm{i}k/F}{1+\mathrm{i}k/F}\right)\exp\left[-2\mathrm{i}\arg\Gamma\left(\frac{1}{4}+\frac{\mathrm{i}k}{4F}\right)\right]. \tag{3.6}$$

From (3.6) we find that

$$|R|^2=1+\mathrm{e}^{-\pi k/F}. \tag{3.7}$$

Furthermore, for $k/F \gg 1$, we use the asymptotic expansion of $\Gamma\left(\tfrac{1}{4}+\mathrm{i}k/4F\right)$ in (3.6) to get

$$\arg R=\frac{k}{2F}\ln\frac{k}{4F}-\frac{k}{2F}-\frac{\pi}{2}+o(1),\quad \frac{k}{F}\gg 1. \tag{3.8}$$

Thus $|R|^2 > 1$, so over-reflection occurs for all values of k. Next we calculate the transmission coefficient T, which is the amplitude of the outgoing wave $\mathrm{e}^{-\mathrm{i}\xi/2}$ at $y=+\infty$ divided by the amplitude of the incoming wave $\mathrm{e}^{\mathrm{i}\xi/2}$ at $y=-\infty$ From (3.1) we have

$$T=\frac{\dfrac{B\Gamma\left(-\tfrac{1}{2}\right)\mathrm{e}^{-\mathrm{i}\pi/8}}{\Gamma\left(-\tfrac{1}{4}+\mathrm{i}k/4F\right)}+\dfrac{A\Gamma\left(\tfrac{5}{2}\right)\mathrm{e}^{5\pi\mathrm{i}/8}}{\Gamma\left(\tfrac{5}{4}+\mathrm{i}k/4F\right)}}{\dfrac{B\Gamma\left(-\tfrac{1}{2}\right)\mathrm{e}^{-\mathrm{i}\pi/8}}{\Gamma\left(-\tfrac{1}{4}+\mathrm{i}k/4F\right)}-\dfrac{A\Gamma\left(\tfrac{5}{2}\right)\mathrm{e}^{5\pi\mathrm{i}/8}}{\Gamma\left(\tfrac{5}{4}+\mathrm{i}k/4F\right)}}. \tag{3.9}$$

By using (3.2) in (3.9) and simplifying the result, we obtain

$$T=\mathrm{i}\mathrm{e}^{-\pi k/2F}. \tag{3.10}$$

The phase of T is just $\pi/2$ and $|T|^2$ is given by

$$|T|^2=\mathrm{e}^{-\pi k/F}. \tag{3.11}$$

From (3.7) and (3.11) we get

$$|R|^2-|T|^2=1. \tag{3.12}$$

This result can be proved directly from (2.2) by means of Green's theorem, as is shown in the Appendix. Graphs of $|R|^2$ and $|T|^2$ as functions of k/F are shown in figure 1. The numerical results of Takehiro & Hayashi (1992), given in their figure 7, are in very good agreement with these exact results.

The results (3.4) and (3.10) were derived on the assumption that k is real and positive, since this assumption was used in deriving (3.1) and in determining the incoming and outgoing waves. However these results express R and T as analytic functions of k, so they can be extended analytically into the whole complex k-plane. They show that both R and T are entire functions of k/F and are independent of c. Therefore there are no discrete modes, since the corresponding eigenvalues would be poles of R or T with respect to k or c. Consequently, the unbounded linear shear flow is linearly stable.

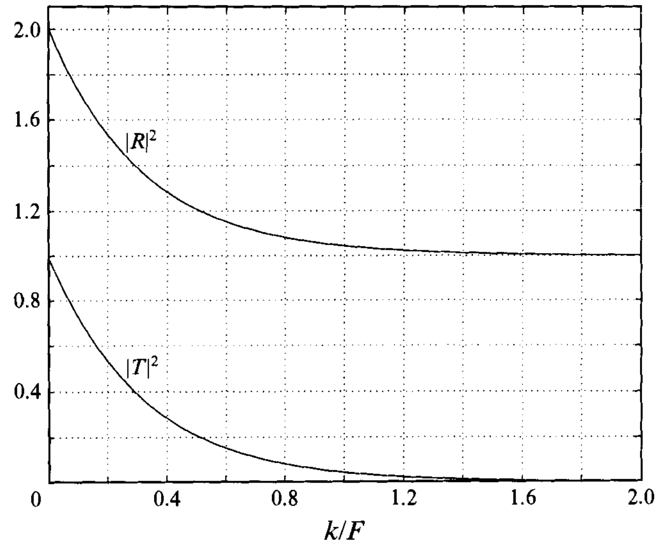

FIGURE 1. Reflection and transmission coefficients for waves incident on an unbounded linear shear flow as functions of the wavenumber k divided by the Froude number F. The ordinate shows $|R|^2 = 1+e^{-\pi k/F}$ and $|T|^2 = e^{-\pi k/F}$, and the abscissa is k/F. Note that $|R|^2 = 1+|T|^2$, that $|R|^2 = 2$ at $k/F = 0$ and $|R|^2 = 1$ at $k/F = \infty$.

4. Modes of a shear flow bounded by a wall

Now we shall consider the reflection of a wave from the same linear shear flow as in the previous section, but confined to the region $y \leqslant y_b$ with a rigid wall at $y = y_b$. The condition that the velocity normal to the wall must vanish leads to the boundary condition

$$h'(y_b) = 0. \tag{4.1}$$

The solution of (2.2) for $h(y)$ is still given by (2.8) for $y \leqslant y_b$, and (4.1) yields the following relation between the constants A and B at $\xi = \xi_b = kF(y_b - c)^2$:

$$A\left\{\xi_b^{3/2}e^{i\xi_b/2}M\left(\frac{5}{4}+\frac{ik}{4F},\frac{5}{2},-i\xi_b\right)\right\}_{\xi_b} + B\left\{e^{i\xi_b/2}M\left(-\frac{1}{4}+\frac{ik}{4F},-\frac{1}{2},-i\xi_b\right)\right\}_{\xi_b} = 0. \tag{4.2}$$

Here and below the subscript ξ_b denotes differentiation with respect to ξ_b. We can satisfy (4.2) by setting $A = C$ times the coefficient of B in (4.2), and setting $B = -C$ times the coefficient of A in (4.2), where C is an arbitrary constant. Then we can write the solution (2.8) for $h(y)$ as follows:

$$h(y) = H(\xi) = C\left\{e^{i\xi_b/2}M\left(-\frac{1}{4}+\frac{ik}{4F},-\frac{1}{2},-i\xi_b\right)\right\}_{\xi_b} \xi^{3/2}e^{i\xi/2}M\left(\frac{5}{4}+\frac{ik}{4F},\frac{5}{2},-i\xi\right)$$
$$- C\left\{\xi_b^{3/2}e^{i\xi_b/2}M\left(\frac{5}{4}+\frac{ik}{4F},\frac{5}{2},-i\xi_b\right)\right\}_{\xi_b} e^{i\xi/2}M\left(-\frac{1}{4}+\frac{ik}{4F},-\frac{1}{2},-i\xi\right). \tag{4.3}$$

To determine the reflection coefficient $R(y_b)$ when a wave is incident from $y = -\infty$, we use the asymptotic form of $h(y)$ for $y \to -\infty$. That asymptotic form is expressed by (3.1) with the appropriate values of A and B, which are contained in (4.3). From (3.1) we have shown that $R(y_b)$ is given by the first expression in (3.3). Into that expression we substitute the ratio A/B determined by (4.2) or (4.3) and we obtain

$$R(y_b) = \left[\frac{\left\{e^{i\xi_b/2}M(-\frac{1}{4}+ik/4F, -\frac{1}{2}, -i\xi_b)\right\}_{\xi_b}\Gamma(\frac{5}{2})e^{-5\pi i/8}}{\Gamma(\frac{5}{4}-ik/4F)} + \frac{\left\{\xi_b^{3/2}e^{i\xi_b/2}M(\frac{5}{4}+ik/4F, \frac{5}{2}, -i\xi_b)\right\}_{\xi_b}\Gamma(-\frac{1}{2})e^{i\pi/8}}{\Gamma(-\frac{1}{4}-ik/4F)}\right] \times \left[\frac{\left\{e^{i\xi_b/2}M(-\frac{1}{4}+ik/4F, -\frac{1}{2}, -i\xi_b)\right\}_{\xi_b}\Gamma(\frac{5}{2})e^{5\pi i/8}}{\Gamma(\frac{5}{4}+ik/4F)} + \frac{\left\{\xi_b^{3/2}e^{i\xi_b/2}M(\frac{5}{4}+ik/4F, \frac{5}{2}, -i\xi_b)\right\}_{\xi_b}\Gamma(-\frac{1}{2})e^{-i\pi/8}}{\Gamma(-\frac{1}{4}+ik/4F)}\right]^{-1}. \tag{4.4}$$

In the Appendix, we prove from the differential equation (2.2) that for c real,

$$|R(y_b)|^2 = 1\,, \qquad c \text{ real.} \tag{4.5}$$

This shows, as we expect, that the wave is completely reflected. This result can also be proved from (4.4).

The eigenvalues of c are the poles of R, which cannot be real as (4.5) shows. Since the numerator in (4.4) is finite for all values of k, F, and ξ_b, the only poles are the zeros of the denominator. By setting the demoninator equal to zero we get an eigenvalue equation for c. We shall solve it when $\xi_b \gg 1$. First we use in it the asymptotic form of M for ξ_b large. This is the same expansion which was used in (3.1) for y large and positive. Then we can write the equation 'denominator of (4.4) = 0' in the asymptotic form

$$-\frac{i}{2}\Gamma\left(-\frac{1}{2}\right)\Gamma\left(\frac{5}{2}\right)\xi_b^{1/4}\left[\alpha\xi_b^{-ik/4F}e^{i\xi_b/2+\pi k/8F} - \frac{2i\xi_b^{ik/4F}e^{-i\xi_b/2+\pi k/8F}}{\Gamma(-\frac{1}{4}+ik/4F)\Gamma(\frac{5}{4}+ik/4F)}\right] = 0. \tag{4.6}$$

Here α is defined by

$$\begin{aligned}\alpha &= \frac{e^{3\pi i/4}}{\Gamma(-\frac{1}{4}-ik/4F)\Gamma(\frac{5}{4}+ik/4F)} + \frac{e^{-3\pi i/4}}{\Gamma(-\frac{1}{4}+ik/4F)\Gamma(\frac{5}{4}-ik/4F)}\\ &= \pi^{-1}e^{3\pi i/4}\sin\left(-\frac{\pi}{4}-\frac{i\pi k}{4F}\right) + \pi^{-1}e^{-3\pi i/4}\sin\left(-\frac{\pi}{4}+\frac{i\pi k}{4F}\right)\\ &= \pi^{-1}e^{\pi k/4F}.\end{aligned} \tag{4.7}$$

We have simplified (4.7) by using the relation just above (3.4).

Now we use (4.7) in (4.6) and rearrange the resulting equation to get the eigenvalue equation

$$e^{i\xi_b}\xi_b^{-ik/2F} = \frac{2\pi i e^{-\pi k/4F}}{\Gamma(\frac{5}{4}+ik/4F)\Gamma(-\frac{1}{4}+ik/4F)}. \tag{4.8}$$

The right-hand side of (4.8) is exactly the complex conjugate of R given by (3.4). It is the reflection coefficient for a wave incident from $y = +\infty$ upon an unbounded linear shear flow. Taking logarithms of both sides of (4.8) yields, with n an integer,

$$i\xi_b - \frac{ik}{2F}\log\xi_b = \log|R| - i\arg R + 2n\pi i. \tag{4.9}$$

We now set $c = c_r + \mathrm{i}c_i$ and then $\xi_b = kF(y_b - c_r - \mathrm{i}c_i)^2 = kF(y_b - c_r)^2 - kFc_i^2 - 2\mathrm{i}kFc_i(y_b - c_r)$. We use this in (4.9), retain only the largest terms assuming that c_i is small, and separate real and imaginary parts to get

$$kc_r = ky_b \pm \left[kF^{-1}(2n\pi - \arg R) + (k^2/2F^2)\log kF(y_b - c_r)^2\right]^{1/2}, \qquad (4.10)$$

$$kc_i = \frac{\log\left(1 + \mathrm{e}^{-\pi k/F}\right)}{4F(y_b - c_r)}. \qquad (4.11)$$

In (4.11) we have used (3.7) for $|R|$. To obtain $\arg R$, which occurs in (4.10), we can use (3.6), or if $k/F \gg 1$ we can use (3.8).

Equations (4.10) and (4.11) yield an infinite sequence of complex eigenvalues of c indexed by the integer n. Equation (4.10) yields both a positive value and a negative value of $y_b - c_r$ for each n. When the positive value is used in (4.11) it yields a positive value of c_i, which corresponds to an unstable mode. Thus there are both a stable and an unstable mode for each n.

Equation (4.11) is of the same form as the 'laser formula' given by Lindzen (1988) in which $F(y_b - c_r)$ is replaced by τ, the time required for a wave to travel from the wall at $y = y_b$ to the turning point at $y = F^{-1}$. That formula was used by Takehiro & Hayashi (1992, equation 44). They obtained R from their numerical calculations for a wave incident upon the unbounded linear shear flow, and they used the group velocity to calculate τ.

5. Modes of a shear flow between a wall and fluid at rest

We next consider a linear shear flow in the finite interval $0 \leqslant y \leqslant y_b$ with a wall at y_b and a state of rest in the region $y \leqslant 0$. Thus we assume that

$$U(y) = y, \quad 0 \leqslant y \leqslant y_b\,; \qquad U(y) = 0, \quad y \leqslant 0\,. \qquad (5.1)$$

Then $h(y)$ still satisfies (2.2) in the interval $0 \leqslant y \leqslant y_b$ with the boundary and continuity conditions

$$h'(y_b) = 0; \qquad h \text{ and } h' \text{ continuous at } y = 0\,. \qquad (5.2)$$

In the region $y < 0$, (2.1) for h simplifies to

$$h'' + k^2(F^2c^2 - 1)h = 0, \qquad y < 0\,. \qquad (5.3)$$

We require that $F|c| > 1$ so that waves of velocity c can propagate in the region $y < 0$.

Now we assume that a wave of unit amplitude is incident on the shear flow from $y = -\infty$, and that it produces a reflected wave of amplitude R. Thus we write the solution of (5.3) for $y < 0$ as

$$h(y) = \mathrm{e}^{-\mathrm{i}k(F^2c^2-1)^{1/2}y} + R\mathrm{e}^{\mathrm{i}k(F^2c^2-1)^{1/2}y}, \qquad y \leqslant 0\,. \qquad (5.4)$$

In the shear flow region $h(y)$ is given by (4.3), which satisfies the differential equation (2.2) and the boundary condition (5.2), i.e. $h'(y_b) = 0$. To satisfy the two continuity conditions in (5.2), we can equate the values of h and h' given by (5.4) at $y = 0$ to the corresponding quantities given by (4.3). Instead we equate the two values of h'/h at $y = 0$, since this eliminates the constant C which occurs in (4.3), and we obtain

$$\mathrm{i}k(F^2c^2 - 1)^{1/2}\frac{1 - R}{1 + R} = \frac{-2kFcN}{D}\,. \qquad (5.5)$$

Here N and D are given by

$$N = \left[\xi_0^{3/2} e^{i\xi_0/2} M\left(\frac{5}{4}+\frac{ik}{4F},\frac{5}{2},-i\xi_0\right)\right]_{\xi_0} \left[e^{i\xi_b/2} M\left(-\frac{1}{4}+\frac{ik}{4F},-\frac{1}{2},-i\xi_b\right)\right]_{\xi_b}$$
$$+\left[e^{i\xi_0/2} M\left(-\frac{1}{4}+\frac{ik}{4F},-\frac{1}{2},-i\xi_0\right)\right]_{\xi_0} \left[\xi_b^{3/2} e^{i\xi_b/2} M\left(\frac{5}{4}+\frac{ik}{4F},\frac{5}{2},-i\xi_b\right)\right]_{\xi_b}, \quad (5.6)$$

$$D = \xi_0^{3/2} e^{i\xi_0/2} M\left(\frac{5}{4}+\frac{ik}{4F},\frac{5}{2},-i\xi_0\right) \left[e^{i\xi_b/2} M\left(-\frac{1}{4}+\frac{ik}{4F},-\frac{1}{2},-i\xi_b\right)\right]_{\xi_b}$$
$$+ e^{i\xi_0/2} M\left(-\frac{1}{4}+\frac{ik}{4F},-\frac{1}{2},-i\xi_0\right) \left[\xi_b^{3/2} e^{i\xi_b/2} M\left(\frac{5}{4}+\frac{ik}{4F},\frac{5}{2},-i\xi_b\right)\right]_{\xi_b}. \quad (5.7)$$

We have written $\xi_0 = kFc^2$ and $\xi_b = kF(y_b - c)^2$.

Upon solving (5.5) for R we get the reflection coefficient

$$R = \frac{-i(F^2c^2-1)^{1/2}D + 2FcN}{-i(F^2c^2-1)^{1/2}D - 2FcN}. \quad (5.8)$$

The proof in the Appendix can be adapted to show that

$$|R| = 1 \text{ for } c \text{ real and } F|c| > 1. \quad (5.9)$$

To find the eigenvalues of c, which are the poles of R, we note that the numerator of (5.8) is finite for c finite. Therefore, the poles are the zeros of the denominator of (5.8), which yields the eigenvalue equation

$$-i(F^2c^2-1)^{1/2}D = 2FcN. \quad (5.10)$$

From (5.9) we see that there are no eigenvalues for real values of c with $|c| > F^{-1}$.

We can simplify (5.10) when k and F are large, with y_b and c fixed, by expanding N and D asymptotically. To do so, we employ the asymptotic expansions of M which are valid in a sector of the ξ-plane containing the positive real axis, since (5.8) was derived on the assumption that ξ_0 and ξ_b are real and positive. When these expansions are used in (5.10) they yield the asymptotic form of the eigenvalue equation:

$$-(F^2c^2-1)^{1/2}\left[(1+e^{-\pi k/F})^{1/2}\sin(\eta_b+\eta_0)+\sin(\eta_b-\eta_0)\right]$$
$$= Fc\left[(1+e^{-\pi k/F})^{1/2}\sin(\eta_b+\eta_0)-\sin(\eta_b-\eta_0)\right]. \quad (5.11)$$

Here η_0, η_b and θ are defined by

$$\eta_0 = \frac{1}{2}\xi_0 - \frac{k}{4F}\log\xi_0 + \frac{\theta}{2} - \frac{\pi}{4},$$
$$\eta_b = \frac{1}{2}\xi_b - \frac{k}{4F}\log\xi_b + \frac{\theta}{2} - \frac{\pi}{4},$$
$$\theta = \arg\left[\Gamma\left(\frac{5}{4}+\frac{ik}{4F}\right)\Gamma\left(-\frac{1}{4}+\frac{ik}{4F}\right)\right]$$
$$= \frac{k}{2F}\log\frac{k}{4F} - \frac{k}{2F} + o(1) \quad \text{as} \quad \frac{k}{F}\to\infty.$$

This problem has been treated numerically by Satomura (1981*a*), Case II). Therefore we shall not solve (5.11) numerically.

6. Modes of a bounded shear flow

Finally we consider the modes of the shear flow $U(y) = y$ in the finite interval $y_a < y < y_b$ with rigid walls at y_a and y_b. Then $h(y)$ satisfies (2.2) in this interval with $h'(y_a) = 0$ and $h'(y_b) = 0$. The solution of (2.2) which satisfies $h'(y_b) = 0$ is given by (4.3). Upon requiring that this solution satisfy $h'(y_a) = 0$ we get the condition

$$\left[\xi_a^{3/2} e^{i\xi_a/2} M\left(\frac{5}{4}+\frac{ik}{F},\frac{5}{2};-i\xi_a\right)\right]_{\xi_a}\left[e^{i\xi_b/2} M\left(-\frac{1}{4}+\frac{ik}{4F},-\frac{1}{2};-i\xi_b\right)\right]_{\xi_b}$$
$$=\left[\xi_b^{3/2} e^{i\xi_b/2} M\left(\frac{5}{4}+\frac{ik}{F},\frac{5}{2};-i\xi_b\right)\right]_{\xi_b}\left[e^{i\xi_a/2} M\left(-\frac{1}{4}+\frac{ik}{4F},-\frac{1}{2};-i\xi_a\right)\right]_{\xi_a}. \quad (6.1)$$

This equation determines the eigenvalues c.

To simplify (6.1) we consider the case in which ξ_a and ξ_b are both large with $y_a < 0 < y_b$. Since $\xi_b = kF(y_b - c)^2$ this means that $\xi_b > 0$ with $\arg(-i\xi_b) = -\pi/2$. Similarly $\xi_a = kF(y_a - c)^2$ so $\xi_a > 0$ with $\arg \xi_a = 2\pi$ so $\arg(-i\xi_a) = 3\pi/2$. We now use the appropriate asymptotic expansions of M in (6.1). Then after a little rearrangement, we can write (6.1) in the asymptotic form

$$2C_+D_+e^{i(\xi_a+\xi_b)/2}(\xi_a\xi_b)^{-ik/4F} + 2C_-D_-e^{-i(\xi_a+\xi_b)/2}(\xi_a\xi_b)^{ik/4F}$$
$$= (C_-D_+ + C_+D_-)\left[e^{i(\xi_a-\xi_b)/2}\left(\frac{\xi_b}{\xi_a}\right)^{ik/4F} + e^{i(\xi_b-\xi_a)/2}\left(\frac{\xi_a}{\xi_b}\right)^{ik/4F}\right]. \quad (6.2)$$

Here $C_\pm$ and $D_\pm$ are defined by

$$C_\pm = \frac{\Gamma(\frac{5}{2})e^{\pi k/8F}e^{\mp 5\pi i/8}}{\Gamma(\frac{5}{4} \mp ik/4F)}, \qquad D_\pm = \frac{\Gamma(-\frac{1}{2})e^{\pi k/8F}e^{\pm i\pi/8}}{\Gamma(-\frac{1}{4} \mp ik/4F)}. \quad (6.3)$$

We now use (6.3) in (6.2) and write the product of gamma functions in the form

$$\Gamma\left(\frac{5}{4}+\frac{ik}{4F}\right)\Gamma\left(-\frac{1}{4}+\frac{ik}{4F}\right) = Ae^{i\theta(k/F)}. \quad (6.4)$$

The phase θ is defined by (5.14) and the amplitude A can be evaluated by using properties of the gamma function (Abramowitz & Stegun 1970, p. 256) with the result

$$A = 2^{1/2}\pi\left(\cosh\frac{\pi k}{2F}\right)^{-1/2}. \quad (6.5)$$

Then we can write (6.2) as

$$\sin\left[\frac{\xi_a}{2} - \frac{k}{4F}\log\xi_a + \frac{\theta(k/F)}{2} - \frac{\pi}{4}\right]\sin\left[\frac{\xi_b}{2} - \frac{k}{4F}\log\xi_b + \frac{\theta(k/F)}{2} - \frac{\pi}{4}\right]$$
$$= \frac{(1+e^{-\pi k/F})^{1/2}-1}{(1+e^{-\pi k/F})^{1/2}+1}\cos\left[\frac{\xi_a}{2} - \frac{k}{4F}\log\xi_a + \frac{\theta(k/F)}{2} - \frac{\pi}{4}\right]$$
$$\times\cos\left[\frac{\xi_b}{2} - \frac{k}{4F}\log\xi_b + \frac{\theta(k/F)}{2} - \frac{\pi}{4}\right]. \quad (6.6)$$

If $k/F \gg 1$, the right side of (6.6) is negligible and therefore either one of the sine functions must be zero. Thus we equate the argument of each sine to a positive integer multiple of π and use the asymptotic form of $\theta(k/F)$ for $k/F \gg 1$. This yields the eigenvalue equations

$$\frac{F}{2}k(y_b-c)^2-\frac{k}{4F}\log\left[(y_b-c)^2F^2\right]+\frac{k}{F}\left(-\frac{1}{4}-\frac{\log 2}{2}\right)=\left(m+\frac{1}{4}\right)\pi\ ,\ m=0,1,\dots\ , \tag{6.7}$$

$$\frac{F}{2}k(y_a-c)^2-\frac{k}{4F}\log\left[(y_a-c)^2F^2\right]+\frac{k}{F}\left(-\frac{1}{4}-\frac{\log 2}{2}\right)=\left(n+\frac{1}{4}\right)\pi\ ,\ n=0,1,\dots\ . \tag{6.8}$$

Equation (6.7) determines $F^2(y_b-c)^2$ as a function of $k/(m+\frac{1}{4})F$ and similarly (6.8) gives $F^2(y_a-c)^2$ as a function of $k/(n+\frac{1}{4})F$. However, to show c as a function of k, we have chosen $y_a=0$ and $y_b=1$, which entails no loss of generality. Then we have calculated c as a function of k for $m=0,1,2,3,4,5$ from each equation for $F=5$ (figure 2*a*) and for $F=7$ (figure 2*b*). These figures are similar to figures 3a and 4a of Satomura (1981*a*). Of course, they are not accurate for small values of k, since they are based on an asymptotic analysis for k large. Equations (6.7) and (6.8) show that $F|y-c|\sim 1.1569$ as $k\to\infty$ where $y=y_a$ or $y=y_b$. This shows that the curves given by (6.7) with $y_b=1$ have the asymptote $c=1-1.1569F^{-1}$ and the curves given by (6.8) with $y_a=0$ have the asymptote $c=1.1569F^{-1}$.

The roots given by (6.7) and (6.8) are real. For each pair of values of n and m, both equations have the same root c for a certain value of k. Thus the curves $c(k)$ given by (6.7) and (6.8) for the pair n,m intersect. Near the intersection points the exponentially small right-hand side of (6.6) must be taken into account, so that (6.7) and (6.8) do not hold there. Instead, there is a small interval of k around each intersection point within which the root c of (6.6) is complex. These intervals and the imaginary part of c within them can be found from (6.6), as in Knessl & Keller (1992). In each interval there are two roots, one with Im $c>0$ and one with Im $c<0$. Since the factor e^{ikct} grows exponentially when $k\text{Im}\ c<0$, the shear flow is unstable to this perturbation. Some of these roots have been found numerically by Satomura (1981*a*).

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Appendix. Derivation of $|R|^2-|T|^2=1$ and $|R(y_b)|^2=1$

To derive (3.12) directly from (2.2) we set $y=y^*+c$ with c real, $h^*(y^*)=h(y)$, and omit the star. Then we multiply (2.2) by $\bar{h}$, the complex conjugate of h, and write the result in the form

$$(y^{-2}\bar{h}h_y)_y-y^{-2}h_y\bar{h}_y+k^2(F^2-y^{-2})h\bar{h}=0. \tag{A1}$$

Next we take the imaginary part of (A1), which comes only from the first term, and integrate it from y_1 to y_2 to get

$$\text{Im}\left[y_1^{-2}\bar{h}(y_1)h_y(y_1)\right]=\text{Im}\left[y_2^{-2}\bar{h}(y_2)h_y(y_2)\right]. \tag{A2}$$

In order to apply (A2) when $y_1<0<y_2$, we must verify that $\frac{d}{dy}\text{Im}[y^{-2}\bar{h}(y)h(y)]$ is integrable about $y=0$. The power series method for y small shows that (2.2) has

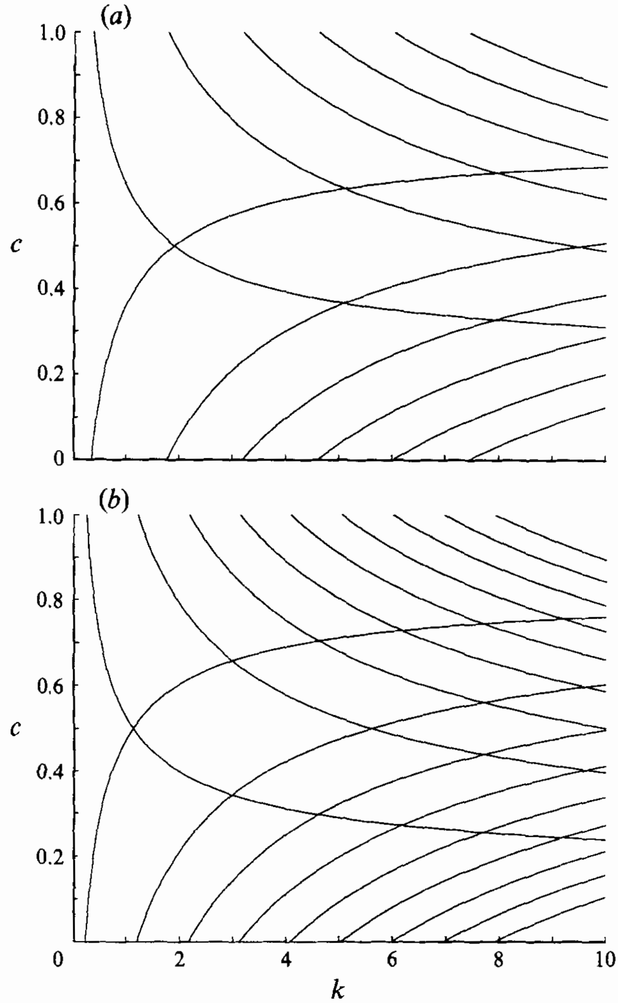

FIGURE 2. The phase velocity c as a function of the wavenumber k for waves on a linear shear flow in a channel. The curves are based upon (6.7) and (6.8) with $y_a = 0, y_b = 1$ and $m = 0, 1, 2, 3, 4, 5$. In both figures the upper curves apply to modes confined near $y_a = 0$ and the lower curves apply to modes confined near $y_b = 1$ and m increases from left to right. Where curves cross there is a short interval in which c is complex. That interval and the value of c in it are determined by (6.6). (*a*) $F = 5$, (*b*) $F = 7$.

one solution of the form $h_1(y) = 1 + \sigma y^2 + O(y^3)$ with σ real, and another solution $h_2(y) = O(y^3)$. For the general solution $h = c_1 h_1 + c_2 h_2$ we have $\bar{h} h_y = 2\sigma |c_1|^2 y + O(y^2)$ so $\mathrm{Im}[\bar{h} h_y] = O(y^2)$. Therefore, $\mathrm{Im}[y^{-2} \bar{h} h_y]$ is finite at $y = 0$, and its derivative is integrable.

Now we use the asymptotic form of $h(y_1)$ for y_1 large and negative and that of $h(y_2)$ for y_2 large and positive in (A2). These asymptotic forms can be obtained by the usual WKB method, and we can write them as follows:

$$\begin{aligned} h(y) = H(\xi) &\sim R\, \xi^{1/4+\mathrm{i}k/4F} \mathrm{e}^{-\mathrm{i}\xi/2} + \xi^{1/4-\mathrm{i}k/4F} \mathrm{e}^{\mathrm{i}\xi/2}, \quad y \to -\infty \\ &\sim T \xi^{1/4-\mathrm{i}k/4F} \mathrm{e}^{-\mathrm{i}\xi/2}, \quad y \to +\infty. \end{aligned} \qquad \text{(A3)}$$

When we use (A3) in (A2) with $y_1 \to -\infty$ and $y_2 \to +\infty$, we obtain (3.11).

We can also use (A2) to treat reflection from a linear shear flow in the range $-\infty < y < y_b$, with a wall at y_b so that $h_y(y_b) = 0$. We use (A2) with $y_2 = y_b$ and then the right-hand side vanishes. Then we let y_1 tend to $-\infty$ and use (A3) for $h(y_1)$. In this way, we get from (A2) the result

$$|R(y_b)|^2 = 1\ , \qquad c \text{ real.} \tag{A4}$$

Thus for c real, which we assumed in writing (A1), we have shown that $|R(y_b)| = 1$. This also proves that there are no poles of R on the real c axis, so there are no purely oscillatory modes.

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